GFAS-Free surface seepage analysis

1. Governing equation of seepage problems. Mathematical formulation

The governing partial differential equation for unconfined seepage flow in the horizontal (x,y) plane is:

$$k_x \frac{\partial^2 \phi}{\partial x^2} + k_y \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1}$$

Where ϕ is the total head which can be expressed also as

$$\phi = y + \frac{p}{\gamma} \tag{2}$$

and where y is the elevation at the point under consideration, p is the fluid pressure, γ is the unit weight of fluid and k_x and k_y are the permeability's in the x and y directions.

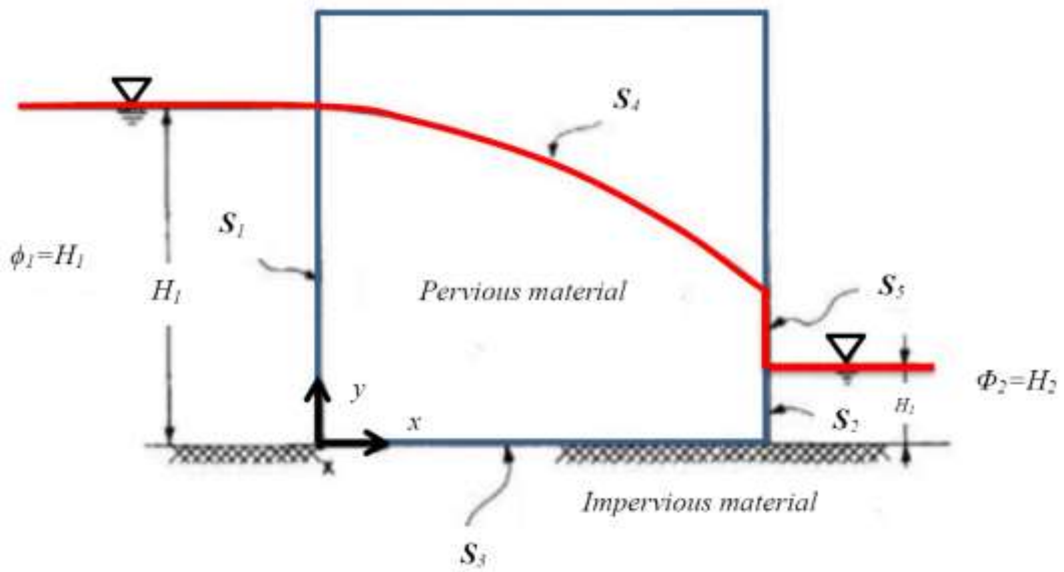


Fig. 1 Unconfined flow, steady state analysis. Boundary conditions.

In the unconfined flow problems as shown in Fig. 1, the exact locations and extend of boundaries S4 and S5 are not known in advance, although the boundary conditions along these surfaces are known. The free surface S4 is characterized by zero pore pressure along the surface with the total head equal to the elevation head. The boundary conditions are [1, 2]:

- For the *upstream* and *downstream* faces:

$$\phi = \phi_1 \text{ on } S_1 \tag{3}$$

$$\phi = \phi_2 \text{ on } S_2 \tag{4}$$

- For the impervious base:

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } S_3 \tag{5}$$

where n denotes the normal to the surface;

- For the free surface

$$\begin{cases} \phi = y \\ \dfrac{\partial \phi}{\partial n} = 0 \end{cases} \text{on } S_4 \qquad (6, 7)$$

- For the surface of seepage:

$$\phi = y \text{ on } S_5 \qquad (8)$$

The finite element discretization process reduces the differential equation (1) to a set of equilibrium type equations of the form:

$$\mathbf{K\Phi} = \mathbf{0}$$
$$\mathbf{K} = \sum_m \iint \mathbf{T}_m{}^T \mathbf{k}_{c_m} \mathbf{T}_m dxdy \qquad (9)$$

Where **K** is the symmetrical permeability matrix of total element assemblage, **Φ** is a vector of all nodal potential (total head) values, $\mathbf{T}_m$ is total potential gradient interpolation matrix of element m, $\mathbf{k}_{cm}$ is the permeability matrix of element m as defined in Eq. (10). Assuming that the principal axes of the permeability tensor coincide with x and y, the property matrix $\mathbf{k}_{cm}$ is:

$$\mathbf{k}_{cm} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \qquad (10)$$

and the **T** matrix is similar with the strain-displacement matrix **B** in the stress analysis. For instance for 4 noded quadrilateral element the matrix **T** is given by:

$$\mathbf{T}_m = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} & \dfrac{\partial N_4}{\partial x} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_4}{\partial y} \end{bmatrix} \qquad (11)$$

And for three noded finite element the matrix T is

$$\mathbf{T}_m = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial x} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_3}{\partial y} \end{bmatrix} \qquad (12)$$

In order to impose the total heads prescribed on the boundaries we add high permeability coefficients to the diagonal elements of **K** corresponding to the boundary nodes, and specify flow conditions that result into the given total potentials. Thus equation (9) is modified to yield:

$$\left(\mathbf{K} + \mathbf{K}^b\right)\mathbf{\Phi} = \mathbf{Q}^b \qquad (13)$$

where $\mathbf{K}^b$ is a diagonal matrix, the ith diagonal element in $\mathbf{K}^b$ is equal to zero if ϕ_i is not prescribed and is otherwise equal to k, where $k >> k_{ii}$ (usually is assumed to be a large value-penalty value defined in GFAS in Steady state flow analysis panel- Fig. 2). Correspondingly, the ith entry in the vector **Q** is equal to zero if ϕ_i is not specified and is otherwise equal to $k\phi_i$.

Considering the natural boundary conditions, it should be noted that the conditions $\dfrac{\partial \phi}{\partial n} = 0$ on S3 and S4 are imposed by not prescribing any flow normal to the surfaces in Equation (9).

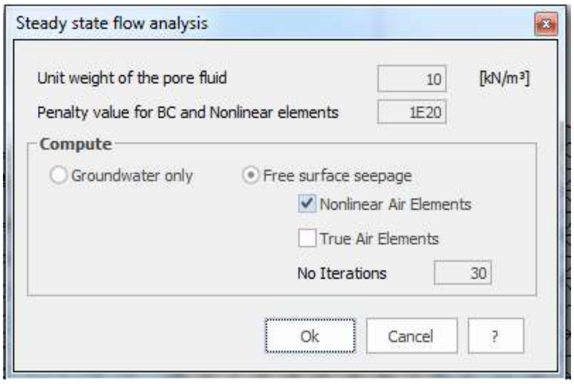


Fig. 2. Steady state flow analysis panel

The solution to the seepage problem could now be obtained if the free surface S4 were known. The finite element discretization of the domain within the boundaries Si (*i*=1…5) would be carried out, and the unknown nodal point total potentials could be solved using equation (13). However, with the location S4 unknown, it is usual practice to assume a free surface, solve equation (13) with not all boundary conditions imposed, check whether all boundary conditions are satisfied and iterate with the free surface S4 until a solution has been obtained which meets all boundary conditions.

The basic requirement in the above finite element mesh iteration solution is that there shall be no flow above the $\phi=y$ line. *This requirement is satisfied by not representing the material above the free surface*. The basis of the scheme presented here is that this requirement can be met more easily computationally by recasting the problem is a non-linear form, in which the natural boundary condition on the free surface in equation (7) is always satisfied (in an integrated sense) and iteration is performed to satisfy also the geometric boundary condition in equation (6).

Assume that the complete soil is represented using a finite element discretization, and let the permeability of the element be:

$$Material\ \ permeability = \begin{cases} k, \ for\,\phi \geq y \\ 0, \ for\,\phi < y \end{cases} \qquad (14)$$

Then the elements above the free surface are effectively removed and those below the free surface are still active. The material permeability in eq. (14) corresponds to a nonlinear permeability which is treated in the GFAS in two different ways either as a *true air elements* (the permeability is set to zero for an element checked to be *totally* above the free surface by considering all of its element nodes and the system of equations are solved applying the Cholesky procedure modified accordingly with the procedure developed in [1]) or as a *nonlinear air element* (the permeability is set to a very low value for an element checked to be *totally* above the free surface by considering all of its element nodes and the system of equations are solved applying the classical Cholesky numerical procedure).

2. Method of analysis

The procedure implemented in GFAS for the solution of free surface seepage problems is summarised as follows:

1. Generate the mesh for the entire domain subjected to the seepage analysis.
2. Determine the skyline profile of the stiffness matrix for the whole mesh, that is, assume the "*true air element*" (*procedure 2*) or "nonlinear air element" (*procedure 1*) to be present in the mesh for all stages of analysis.
3. Evaluate the system stiffness matrix, introduction of the prescribed boundary conditions and solve Eq. (13) for the nodal total head.
4. Any point which is above the free surface will have the total head exceeding the elevation head. If an element is checked to be *totally* above the free surface by considering its entire element nodes, it will be assigned as "true air element" or "nonlinear element" and will be neglected in all the later computations. Any element which is below or partially below the free surface during the iteration will not be deleted from the analysis.
5. Determine the new system stiffness matrix with the original skyline profile as determined in Step 2. The "*true air elements*" will not enter into any computation. The "nonlinear elements" will be considered but with a low value for permeability.
6. Repeat steps 5 to 6 until there is no further "air node" or "nonlinear element" being generated form the iteration.
7. If external free surface which is given by S_5 in Fig. 1 exists, the unknown location of the external free surface can be found by gradually turning the external nodes to nodes with prescribed head from the lowest level upward and repeat steps 5 to 7.
8. Convergence is achieved if there is no air element of air node generated during iteration and all the external free surface nodes satisfy the boundary conditions.
9. The precise location of the free surface and equipotential lines can be determined by an interpolation process.

3. Computational example

Let us consider am earth dam with sloping sides and a relatively impermeable clay core as presented in Fig. 3.

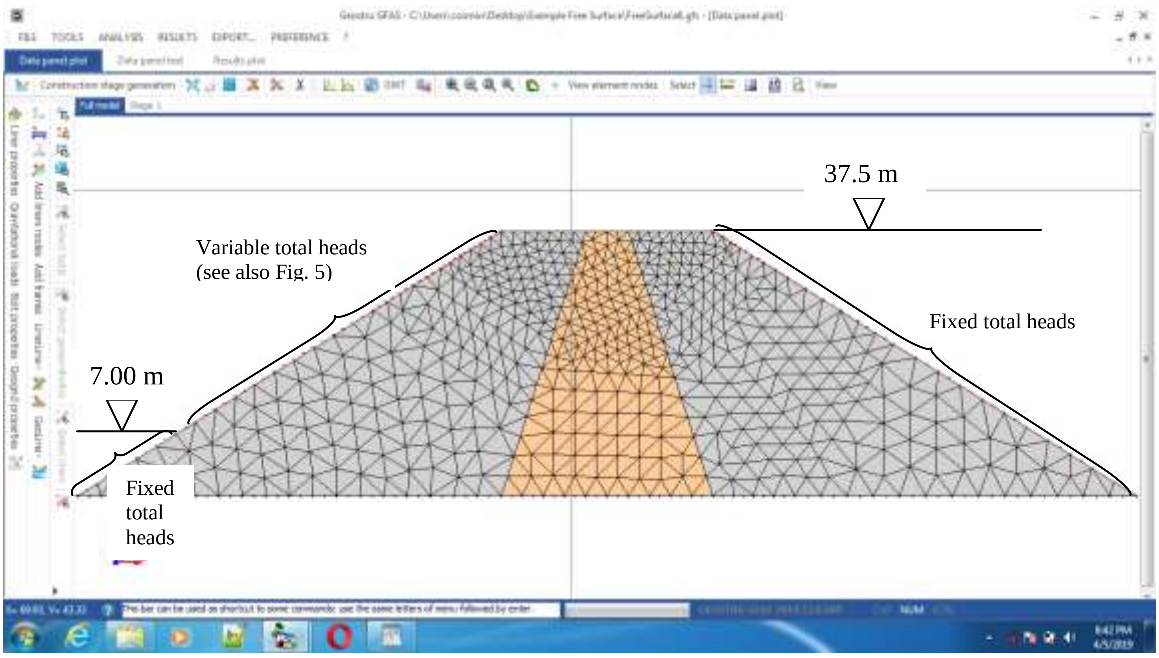


Fig. 3. Earth dam with sloping sides

The horizontal free surface upstream is set at an elevation of 37.5 m (Fig. 3). The nodes on the upstream face of the dam are also set at a total head of 37.5 m while on the downstream face the nodes up to the elevation of 7m are fixed at a total head of 7.0m (Fig. 4).

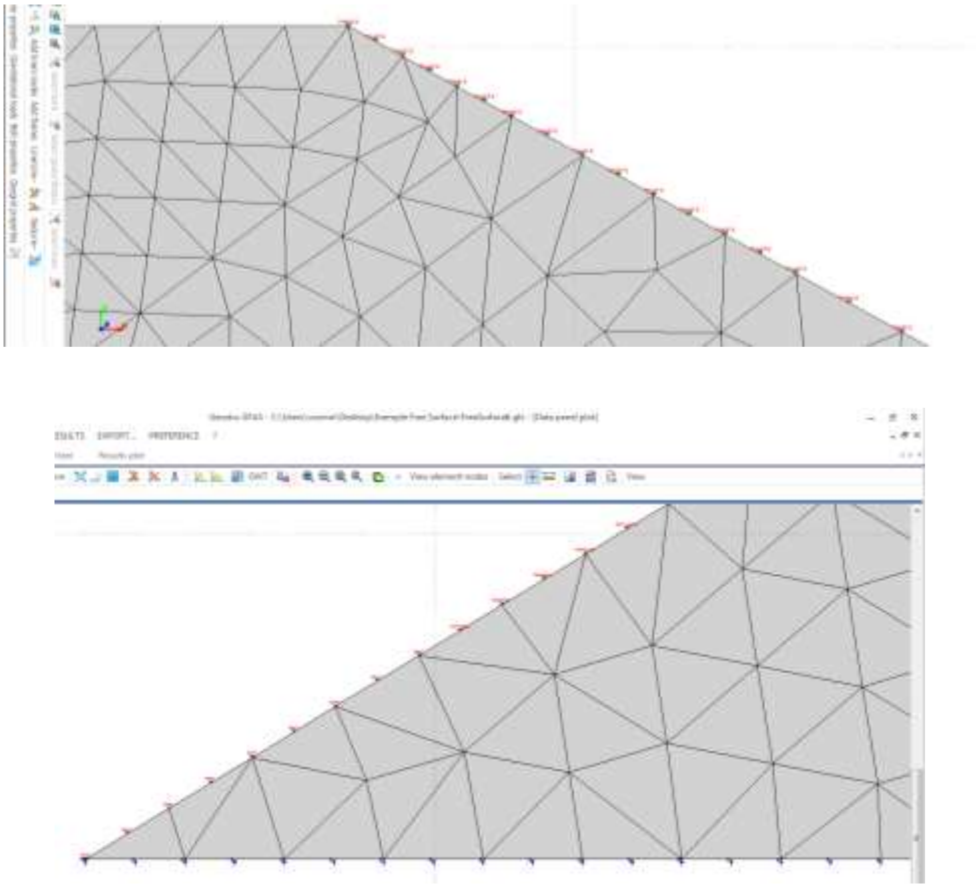


Fig. 4. Imposing the fixed total heads: (a) upstream and (b) downstream

The rest of the nodes on the downstream face are set to a variable total head equal with the elevation at that points (Fig. 5), detecting in this way the existence of external free surface (boundary S_5 see Fig. 1).

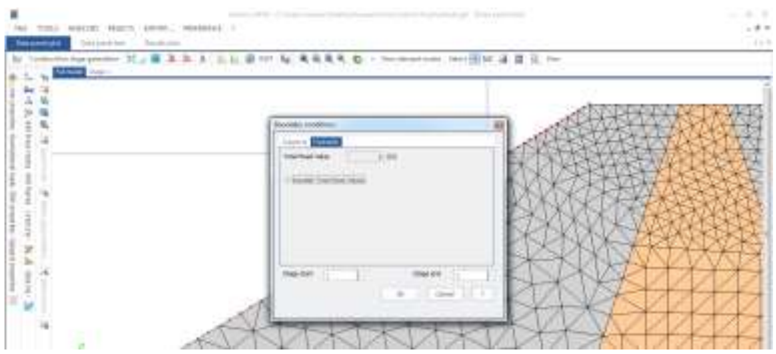


Fig. 5. Imposing the variable total heads.

The hydraulic properties (permeability) are shown in Fig. 6.

Fig. 6. Hydraulic properties.

The analysis is launched following the panel data defined in Fig. 7. The procedure 1 corresponds to the option “Nonlinear air elements” whereas the procedure 2 corresponds to the option “True air elements” (see also the section 2 Method of analysis of the present document).

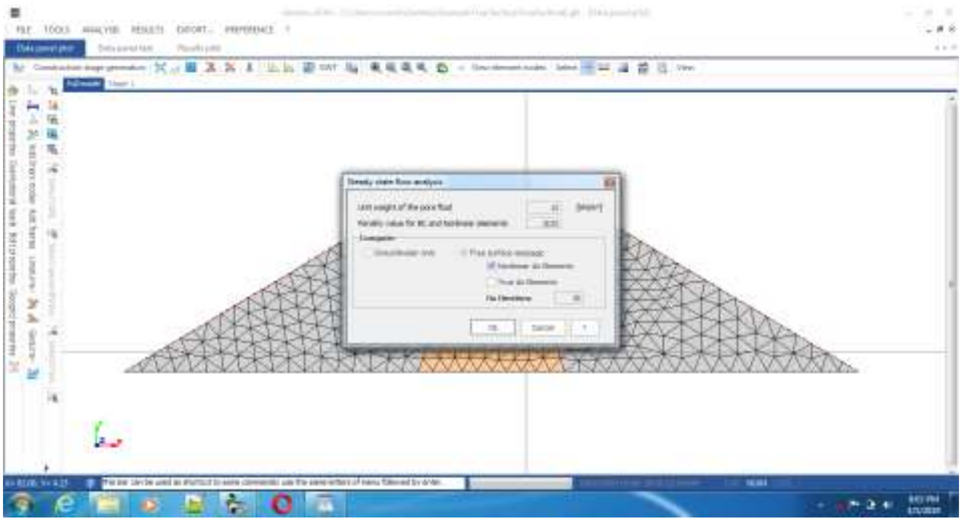
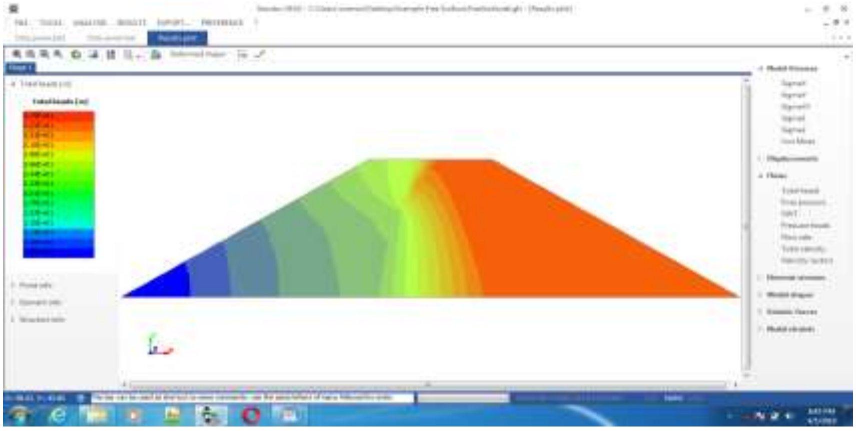
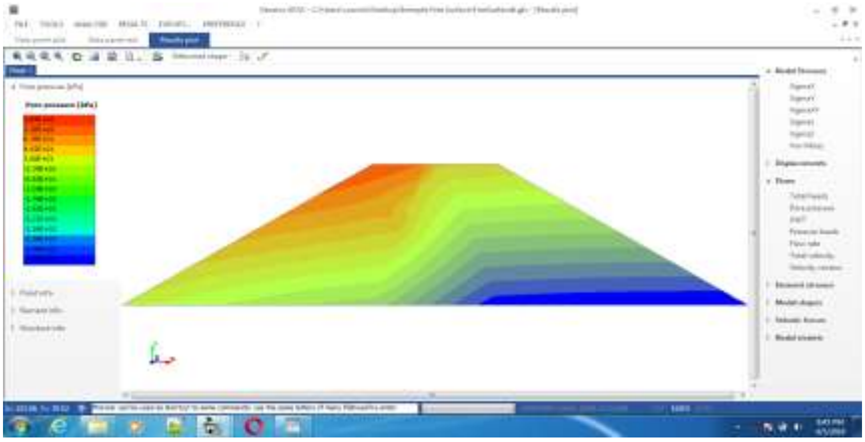
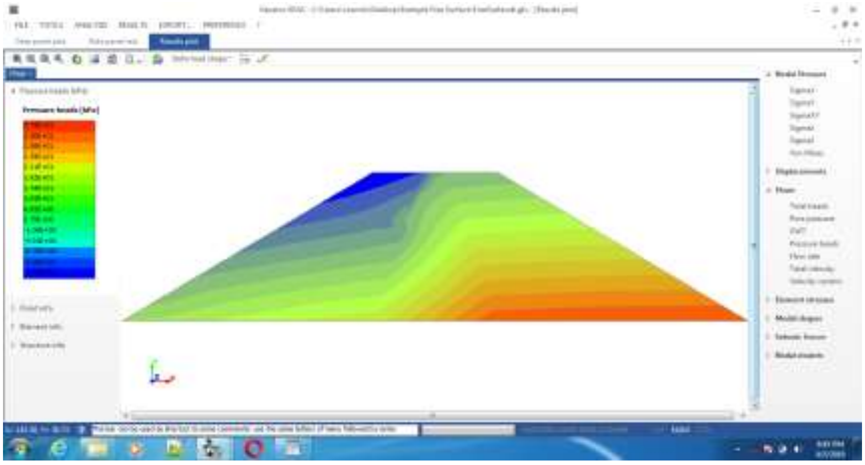


Fig. 7. Steady seepage analysis panel.

The results are presented graphically in terms of Total heads, Pore pressures, Pressure heads, Total velocity and Velocity vectors (Fig. 8).







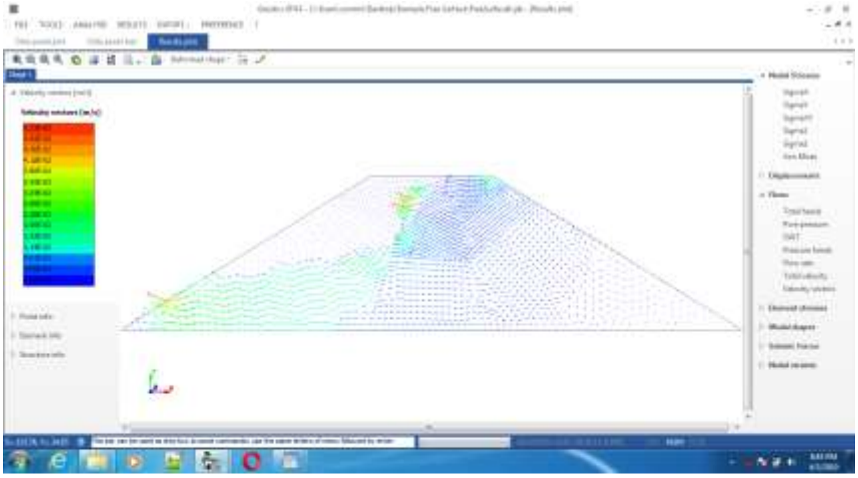


Fig. 8. Graphical output.

The user may see the free surface (obtained as already mention by interpolation) setting on the panel Plot->Isolines->Zero level value as is depicted in Fig. 9 and then showing the distribution of Pressure heads (Fig. 10).

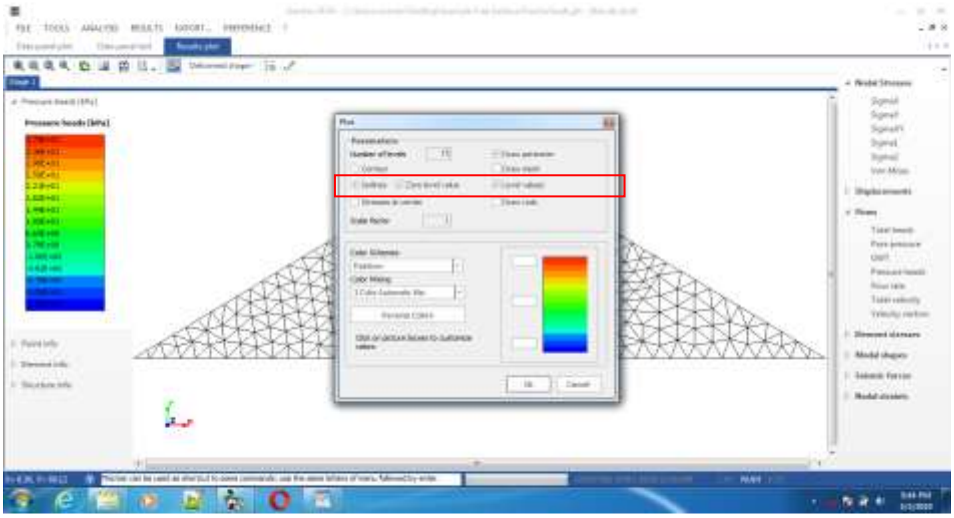


Fig. 9. Plot set-up for free surface line

Fig. 10. Free surface representation.

4. References

1. Cheng, Y.M., Tsui Y., An efficient method for the free surface seepage flow problems, Computers and Geotechnics, vol 15., pp. 47-62, 1993.
2. Bathe, K.J., Khoshgoftaar, M.R., Finite element free surface seepage analysis without mesh iteration, International Journal for Numerical and Analytical Methods in Geomechanics, Vol.3, pp. 13-22, 1979.